INTRODUCTION TO THE PROBABILISTIC QFT

J. J. Martín-Romero

Electronic Department, IES M.B.Cossio. C/ Manuel Bartolomé, 6, 26200 Haro (La Rioja), SPAIN

A. Martín-Romero

Computer Department, IES Rio Arba. Av. del Pilar, s/n, 50660 Tauste (Zaragoza), SPAIN

Abstract: The study of uncertainty control systems by means of QFT suppose that probability distribution of parameters is uniform and the controller is designed to work in any parameter value combination, to get complete robustness. The complete robustness is impossible, for this reason, in this paper the problem is seen from different angle. It is outlined the possibility of eliminating some parameters values combination, if the probability for this combinations is low. Thus, in most systems the parameter probability is not uniform, for this reason it is studied the consequence of supposing other distribution of probability, in particular beta distribution. Finally, the repercussion over bounds is studied too.

Keywords: Quantitative Feedback Theory (QFT), probability, beta.

1. INTRODUCTION

QFT works with uncertain plants to approximate real world to theory. Imagine a plant whose behaviour can be described by,

$$P(s) = \frac{k}{s^3 + as^2 + bs + c} \tag{1}$$

During system identification is observed that *k* and *c* always are 10 and 20 respectively; but *a* and *b* parameters value can be very different. It depends on internal and external plant factors, and the values are: $a \in [3, 8]$ and $b \in [12, 22]$. The present way to interpret the possibility of different values is that these can have any value randomly. It is equivalent to say that the values of parameters have uniform probability.

It is true in some cases, but false in others. For example, the nominal value of a resistence, the value of H_{FE} in bipolar transistor, the electrical and mechanical time constant of DC motor, etc. In these cases is more likely a value around some than others (always into specified interval).

The feedback cost is increased due to supposing that every parameter combination has the same probability. This cost can be reduced if a low design error is assumed (it is an error in case the model was perfect, as it is impossible a perfect model the error always occurs).

Nowdays a study of QFT probability has not been done, but it has been observed that to apply some techniques is convenient to know the probability distribution of uncertainty parameters. For example in a recent García-Sanz paper is exposed the necessity of choosing a probable plant to obtain the correct design of multivariable controller.

2. TO MAKE ROUND THE TEMPLATE CORNERS AND ERROR PROBABILYT IF IT WORKS WITH UNIFORM DISTRIBUTION.

The QFT design is based on the templates. Every template is the contour of uncertainty surface on Nichols chart of a plant working to one frequency.

Template is the projection of n-dimensional parameter space over complex plane by means of plant, P.

$$P:\mathbb{R}^n \rightarrow \mathbb{C} \qquad \vee \qquad P:\mathbb{R}^n \rightarrow \mathbb{R}^2 \qquad (2)$$

For (1) the problem would be to move a two dimension space (two uncertainty parameters) into other two dimension space by means of plant transfer function. In this case a surface is transformed into other surface. We suppose that the contour of parameters space is transformed in the contour of template on Nichols chart, as it is shown in figure 1 (although it has found some case where this is not true, García-Sanz and Vital. In any case it has not been found any plant where corners of parameter space are not transformed in corners on Nichols chart).

If uncertainty area in parameter space grow, the template at frequency ω will grow too. And, if template is large, the QFT bound will be very strict and of course to find the controller will be more difficult and with greater feedback cost.

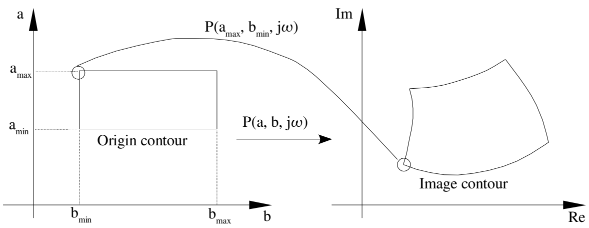

Figure 1.- Conversion of origin contour in image contour example.

The macroscopic world, where control theory is applied, is continuous; and it is difficult to find some abrupt behaviour. For this reason is hard to believe in a parameter space bounded by straight lines or abrupt figures (like rectangles). This justifies, in a qualitative sense, the abrupt boundaries rejection, such as rectangles, cuboids or hypercuboids. It is easier believe in rounded boundaries.

The figure 2 shows the parameter space when we make corners round. The uncertainty surface is reduced and the QFT bound can be less strict. If it is supposed that the non rounded contour describes the uncertainty perfectly, an error will be made due to the elimination of some combinations of uncertainty parameters. A way to measure this error can be to calculate the occurrence probability of these combinations.

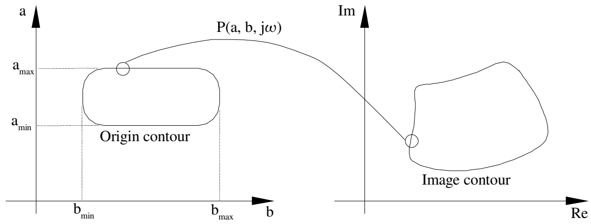

Figure 2.- Conversion of origin rounded contour in image contour example.

For example in (1), if i is supposed that every value of uncertainty intervals, $[a_{min}, a_{max}]$ and $[b_{min}, b_{max}]$, has the same probability of occur, the probability distributions would be uniform continuous (like figure 3).

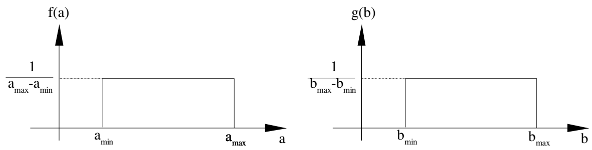

Figure 3.- Uniform probability distribution for *a* and *b* parameters.

In this case, the probability of *a* to have values between a_i and a_j is calculated as,

$$Pr(a_{i-j}) = \int_{a_i}^{a_j} f(a)\, da = \\ = \int_{a_i}^{a_j} \frac{1}{a_{max}-a_{min}}\, da = \frac{a_j - a_i}{a_{max}-a_{min}} \qquad (3)$$

An analogous expression can be obtained for *b* parameter. The plant probability to work with $a \in [a_i, a_j]$ and $b \in [b_i, b_j]$ at same time will be

$$Pr(i-j) = Pr(a_{i-j}) \cdot Pr(b_{i-j}) = \\ = \frac{a_j - a_i}{a_{max}-a_{min}} \cdot \frac{b_j - b_i}{b_{max}-b_{min}} = \frac{Error\ area}{Total\ area} \qquad (4)$$

Where *Error area* is the eliminated surface by roundig corners. Maybe, the easier way to make round corners is to use a circumference of radius *r*, tangent to edges (see figure 4). The *error area* is $(4-\pi)\cdot r^2$, this means that error probability is,

$$Pr = \frac{(4-\pi)\, r^2}{(a_{max}-a_{min})(b_{max}-b_{min})} \qquad (5)$$

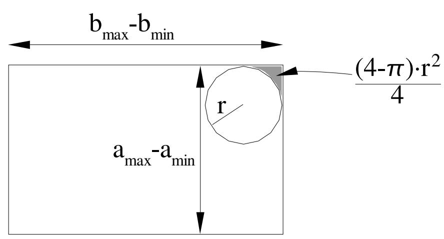

Figure 4.- Despised area in each corner if it is rounded by means of circumference.

If it is defined k like maximum proportional relationship between two rectangle edges and r_{max}, the maximum possible radius, a table with several probable errors can be built.

$$k=\frac{max\left(b_{max}-b_{min},a_{max}-a_{min}\right)}{min\left(b_{max}-b_{min},a_{max}-a_{min}\right)}$$
$$r_{max}=min\left(\frac{b_{max}-b_{min}}{2},\frac{a_{max}-a_{min}}{2}\right) \tag{6}$$

$$Pr=\frac{(4-\pi)r^2}{4kr_{max}} \tag{7}$$

Where Pr is the probable error.

		Pr (%)		
r	k	k = 1	k = 1.5	k = 2
$0.1\,r_{max}$	$\frac{(4-\pi)}{4k}$	0.214	0.143	0.107
$0.2\,r_{max}$	$\frac{4(4-\pi)}{4k}$	0.858	0.572	0.429
$0.5\,r_{max}$	$\frac{25(4-\pi)}{4k}$	5.365	3.576	2.682
$0.7\,r_{max}$	$\frac{49(4-\pi)}{4k}$	10.515	7.010	5.152
r_{max}	$\frac{100(4-\pi)}{4k}$	21.460	14.306	10.515

Table 1.- Relative error probability if uncertainty rectangle in parameters space is rounded.

If an adequate solution is chosen the rounded space can be a good approximation.

The problem was proposed with two uncertainty parameters. If we work with three uncertainty parameters, we will make them round with spheres and cylinders. With n uncertainty parameters we will use hyperspheres and hypercylinders of n dimensions. It is true that proposed solution can get meaningless, but this problem can be solve as it is indicated in the next section.

3. TO MAKE ROUND THE TEMPLATE IN SPACE PARAMETERS.

When plant has one uncertainty parameter, the parameter space is a segment. When it has two the parameter space is a plane surface, normally a rectangle. When it has three it is had a cuboid in parameter space. For n parameters, the parameter space is an hypercuboid.

If we want to round these ones, we will find problems due to the necessity of interconnect hypercylinders, hyperspheres and hyperplanes. It is better to use a figure that allow to represent the parameters space in a simple form. Here arises the function,

$$\begin{aligned}(a_1p_1-x_{1c})^{m_1}+(a_2p_2-x_{2c})^{m_2}+...\\ ...+(a_np_n-x_{nc})^{m_n}-1=0\end{aligned} \tag{8}$$

It allows to build a rounded hypercuboid. Where p_i represents the uncertainty parameter i. a_i indicates the length of edge hypercuboid in the dimension-i by means of equation,

$$L_i=\frac{2}{a_i} \tag{9}$$

The x_{ic} variables allow to calculate the mass centre:

$$p_{mc}=\left(\frac{x_{1c}}{a_1},\frac{x_{2c}}{a_2},...,\frac{x_{nc}}{a_n}\right) \tag{10}$$

Finally, m_i make round more or less depending its value, it must be even number. To understand this, imagine the two dimensions case.

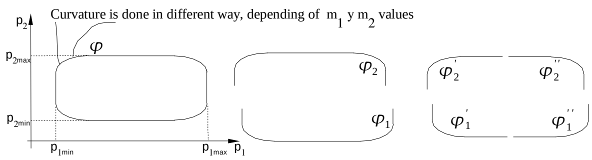

Figure 5.- Rounded rectangle created by (8). Parts of rounded template.

Figure 5 shows the rounded template (and its parts) created by $(a_1p_1-x_{1c})^{m_1}+(a_2p_2-x_{2c})^{m_2}-1=0$. This curve is called φ. Curvature of x axis is different from y axis one. But in the four corners is the same. This can be changed due to,

$$\varphi_1=\frac{-\sqrt[m_{1a}]{1-(a_2p_2-x_{2c})^{m_{2a}}}+x_{1c}}{a_1} \tag{11}$$

$$\varphi_2=\frac{\sqrt[m_{1b}]{1-(a_2p_2-x_{2c})^{m_{2b}}}+x_{1c}}{a_1} \tag{12}$$

And both can be divided into other two ones, like is shown in figure 5.

$$\begin{aligned}
\varphi_1^{'} &= \frac{-\sqrt[m_{1a}^{'}]{1-(a_2 p_2 - x_{2c})^{m_{2a}^{'}}} + x_{1c}}{a_1}\\
\varphi_1^{''} &= \frac{-\sqrt[m_{1a}^{''}]{1-(a_2 p_2 - x_{2c})^{m_{2a}^{''}}} + x_{1c}}{a_1}\\
\varphi_2^{'} &= \frac{\sqrt[m_{1b}^{'}]{1-(a_2 p_2 - x_{2c})^{m_{2b}^{'}}} + x_{1c}}{a_1}\\
\varphi_2^{''} &= \frac{\sqrt[m_{1b}^{''}]{1-(a_2 p_2 - x_{2c})^{m_{2b}^{''}}} + x_{1c}}{a_1}
\end{aligned} \tag{13}$$

To obtain a_1, a_2, x_{1c} and x_{2c}, the figure 6 can be seen. It is shown relationship between variables, dimensions and position on parameter space. By analogy it is easy to obtain relationship for n dimensions.

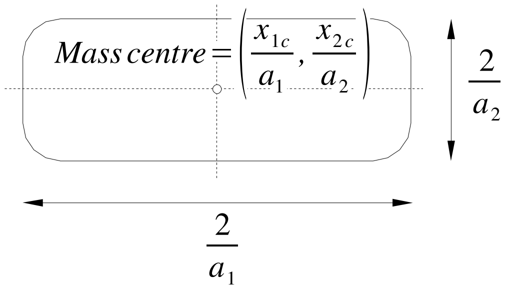

Figure 6.- Relationship between rounded rectangle and $(a_1 p_1 - x_{1c})^{m_1} + (a_2 p_2 - x_{2c})^{m_2} - 1 = 0$

How is it used (8) to calculate error probability? In the case of two uncertainty parameters is easy, only it is necessary to solve,

$$p_{error} = 1 - \int_{p_{1min}}^{p_{1max}} \left(\int_{\frac{-\sqrt[m_2]{1-(a_1 p_1 - x_{1c})^{m_1}} + x_{2c}}{a_2}}^{\frac{\sqrt[m_2]{1-(a_1 p_1 - x_{1c})^{m_1}} + x_{2c}}{a_2}} P(p_1, p_2)\, dp_2 \right) dp_1 \tag{14}$$

Where $P(p_1,p_2)$ is the probability density function of p_1 and p_2 uncertainty parameters.

Note that when uncertainty parameters are two, (14) has solution, but if uncertainty parameters number is high, (14) can not be solved. In case of n uncertainty parameters P_{error} would be,

$$\begin{aligned}
p_{error} = 1 - \int_{a_{1min}}^{a_{1max}} \Bigg(\int_{limit2_{min}}^{limit2_{max}} \Bigg(\int_{limit3_{min}}^{limit3_{max}} \ldots \\
\ldots \Bigg(\int_{limitn_{min}}^{limitn_{max}} P(p_1, p_{2,\ldots}, p_n)\, dp_n \Bigg) \ldots dp_3 \Bigg) dp_2 \Bigg) dp_1
\end{aligned} \tag{15}$$

where,

$$lim2_{min} = \frac{-\sqrt[m_2]{1-(a_1 p_1 - x_{1c})^{m_1}} + x_{2c}}{a_2}$$

$$lim2_{max} = \frac{\sqrt[m_2]{1-(a_1 p_1 - x_{1c})^{m_1}} + x_{2c}}{a_2}$$

$$lim3_{min} = \frac{-\sqrt[m_3]{1-(a_1 p_1 - x_{1c})^{m_1} - (a_2 p_2 - x_{2c})^{m_2}} + x_{3c}}{a_3}$$

$$lim3_{max} = \frac{\sqrt[m_3]{1-(a_1 p_1 - x_{1c})^{m_1} - (a_2 p_2 - x_{2c})^{m_2}} + x_{3c}}{a_3}$$

$$\vdots$$

$$limn_{min} = \frac{-\sqrt[m_n]{1-(a_1 p_1 - x_{1c})^{m_1} - (a_2 p_2 - x_{2c})^{m_2} - \ldots}}{a_n}$$
$$\frac{\ldots - (a_{n-1} p_{n-1} - x_{(n-1)c})^{m_{n-1}} + x_{nc}}{}$$

$$limn_{max} = \frac{\sqrt[m_n]{1-(a_1 p_1 - x_{1c})^{m_1} - (a_2 p_2 - x_{2c})^{m_2} - \ldots}}{a_n}$$
$$\frac{\ldots - (a_{n-1} p_{n-1} - x_{(n-1)c})^{m_{n-1}} + x_{nc}}{}$$

The result is an integral that is impossible to solve. Then, why must we use (8)? Because we can obtain in a simple way (doing derivative) the director vectors of planes or hiperplanes that get the linearity surface. The complexity of one integral is changed by several simple integrals. For example, in a plant with three uncertainty parameters (figure 7), we must integrate the different pieces instead of a complex function.

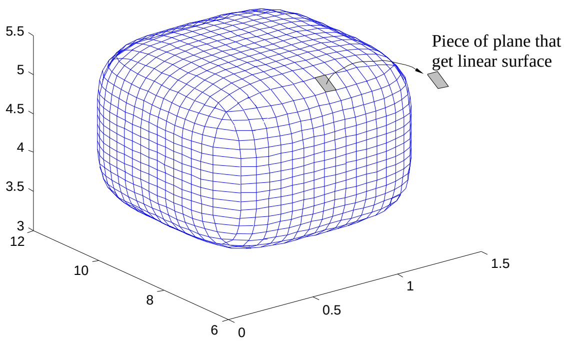

Figure 7.- Rounded space of three uncertainty parameters divided in small pieces of plane.

4. ERROR PROBABILITY IF IT IS SUPPOSED BETA DISTRIBUTION.

It is logical to think that some parameter values have more probability of occurring than others. The probability distribution will not be continuous-uniform. How can we approximate the real probability distribution? A good solution can be beta distribution. This one is enclosed between two values and its shape can be chosen. In figure 8, it is shown a beta distributon example, with one uncertainty parameter $a \in [a_{min}, a_{max}]$.

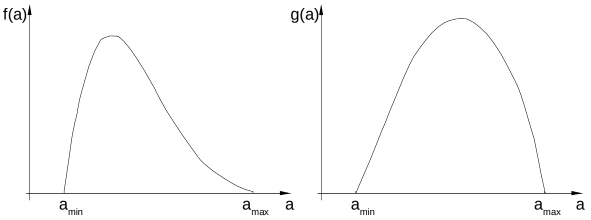

Figure 8.- Beta distribution examples.

The density function, *f(a)*, of a random parameter $a \in [a_{min}, a_{max}]$, is

$$\begin{cases} f(a)=0 & a \leqslant a_{min} \\ f(a)=k(a-a_{min})^{\alpha}\,(a_{max}-a)^{\beta} & a_{min}<a<a_{max} \\ f(a)=0 & a \geqslant a_{max} \end{cases} \quad (16)$$

k is a constant that depends on a_{min}, a_{max}, α and β, it is chosen to get unity area between a_{min} and a_{max}. The α and β indicate the asymmetry. Immediately it is noted that round corners with beta distribution implicates a minor error than with uniform distribution.

Now it's going to be supposed a system with two uncertainty parameters, whose beta distribution are,

$$\begin{cases} f(a)=k_1(a-a_{min})^{\alpha_1}\,(a_{max}-a)^{\beta_1} & a_{min}<a<a_{max} \\ g(b)=k_2(b-b_{min})^{\alpha_2}\,(b_{max}-b)^{\beta_2} & b_{min}<b<b_{max} \end{cases} \quad (17)$$

When it is supposed independence between parameters, it is possible to built the density distribution of both,

$$\begin{aligned} &P(a,b)= \\ &=k_1 k_2(a-a_{min})^{\alpha_1}\,(a_{max}-a)^{\beta_1}(b-b_{min})^{\alpha_2}\,(b_{max}-b)^{\beta_2} \end{aligned} \quad (18)$$

An example of this density function is shown in figure 9, where logically, enclosed volume will be 1.

ab plane is the parameter space. If the uncertainty rectangle is changed by a rounded uncertainty rectangle, the error probability can be calculated. This is shown in figure 10.

First the rounded curve must be represented. This one is called regular enclosed domain, D. The probability of the plant to get some a, b parameters combination

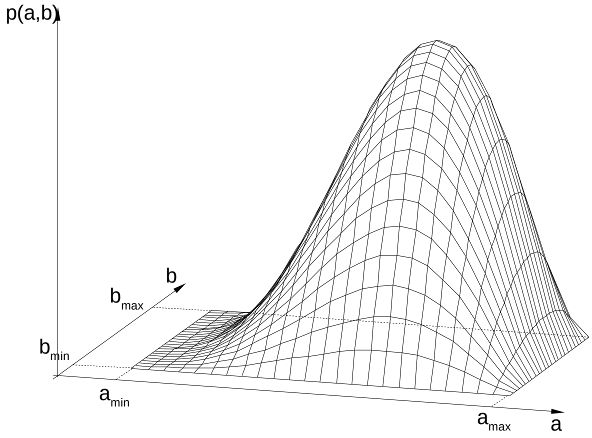

Figure 9.- Beta distribution of two parameters.

placed outside rounded contour could be calculated by means of,

$$\begin{aligned} p_{error} &= 1-\iint_D p(a,b)\,db\,da= \\ &=1-\int_{a_{min}}^{a_{max}}\left(\int_{\varphi_1(a)}^{\varphi_2(a)} P(a,b)\,db\right)da \end{aligned} \quad (19)$$

Where φ_1 and φ_2 are two curves (hypersurfaces in case of n parameters) that define the rounded uncertainty contour (see figure 10) and they arise from $(w\,a-x_c)^n+(h\,b-y_c)^n-1=0$. P(a,b) is (18) expresion.

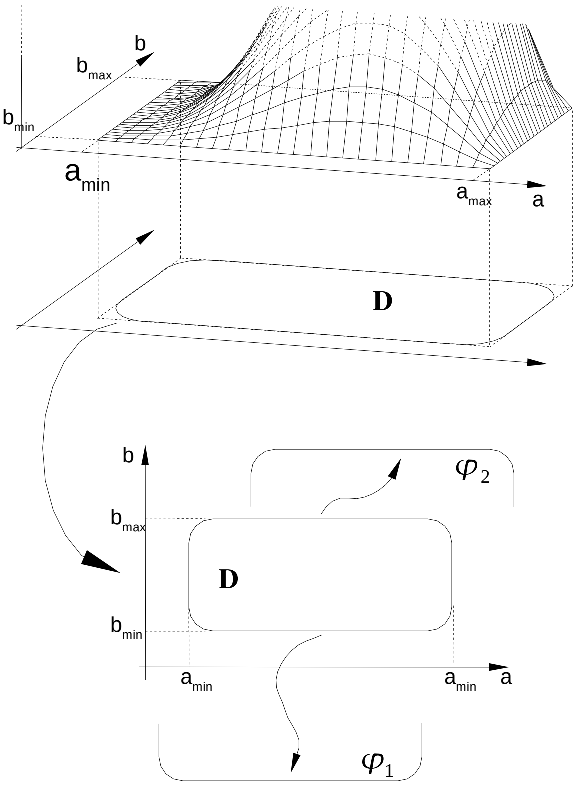

Figure 10.- Rounded domain description in parameter space.

If we call, Ψ(a), to the integral,

$$\Psi(a)=\int_{\frac{-\sqrt[n]{1-(w\,a-x_c)^n}+y_c}{h}}^{\frac{\sqrt[n]{1-(w\,a-x_c)^n}+y_c}{h}} P(a,b)\,db \qquad (20)$$

and is solved, the solution is

$$\Psi(a)=\frac{k_1 k_2\left(a-a_{min}\right)^{\alpha_1}\left(a_{max}-a\right)^{\beta_1}}{(1+\alpha_2)h}$$

$$\left(h\left(b_{max}-b_{min}\right)^{\beta_2}\left(V_2(a)-b_{min}\right)^{1+\alpha_2}\; HG_1 \right. \qquad (21)$$

$$\left. -h\left(b_{max}-b_{min}\right)^{\beta_2}\left(V_1(a)-b_{min}\right)^{1+\alpha_2}\; HG_2\right)$$

Where,

$$V_1(a)=\frac{\left(1-\left(a\,w-x_c\right)^n\right)^{\frac{1}{n}}-y_c}{h}$$

$$V_2(a)=\frac{\left(1-\left(a\,w-x_c\right)^n\right)^{\frac{1}{n}}+y_c}{h}$$

and HG_1 y HG_2 are hypergeometric series

$$HG_1={}_2F_1\left[1+\alpha_{2,}-\beta_{2,}2+\alpha_2;-\frac{b_{min}-V_2(a)}{b_{max}-b_{min}}\right]$$

$$HG_2={}_2F_1\left[1+\alpha_{2,}-\beta_{2,}2+\alpha_2;-\frac{b_{min}+V_1(a)}{b_{max}-b_{min}}\right]$$

with

$${}_2F_1(a,b,c;z)=1+\frac{a\,b}{c}z+\frac{a(a+1)b(b+1)}{c(c+1)2!}z^2+\ldots$$

The obtained results have complexity, this makes impossible to solve (19) if uncertainty parameters is high.

5. EXAMPLE.

Suppose we have a plant with two uncertainty parameters $a\in[3,\ 8]$ and $b\in[7,12]$. The probabilistic density function of both are:

$$f_{beta}(a)=0.022018(a-3)^{0.7}\,(8-a)^2$$

$$g_{beta}(b)=0.004303(b-7)\,(22-b)^{0.5}$$

we want to find the error if we make round the uncertainty rectangle in different ways.

The beta distributions are,

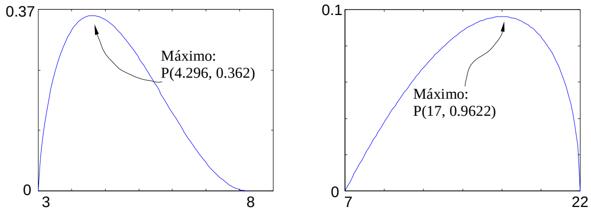

Figure 11.- Beta distributions of $f_{beta}(a)$ and $g_{beta}(b)$.

Then P(a,b) would be,

$$p_{beta}(a,b)=$$
$$=9.475\ 10^{-5}(a-3)^{0.7}(8-a)^2(b-7)\,(22-b)^{0.5}$$

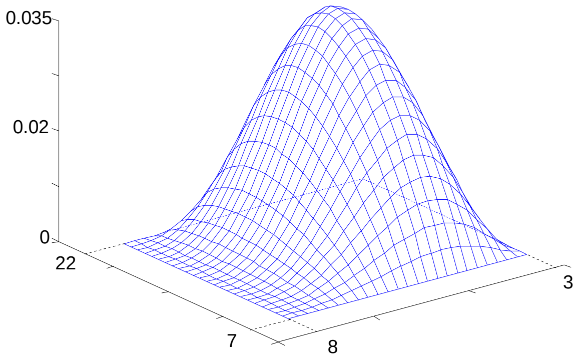

Figure 12.- $P_{beta}(a,b)$ distribution graphic.

Next step is to calculate the rounded rectangle. Using figure 6 is inferred $x_c=2.2$, $y_c=3.8$, $w=0.4$ and $h=0.4$. It is chosen $m_1=m_2=n$. Finally the expresion

$$p_{error}=1-\int_3^8\left(\int_{-7.5\sqrt[n]{1-(0.4\,a-2.2)^n}+14.5}^{7.5\sqrt[n]{1-(0.4\,a-2.2)^n}+14.5} 9.475\ 10^{-5}\left(a-3\right)^{0.7}\left(8-a\right)^2\left(b-7\right)\left(22-b\right)^{0.5} db\right)da$$

must be evaluated.

If n is high then we make round the corners less than if n is low. So, error probability is minor if n is a high number. The results for different n values are

n	**p**
4	1.40186%
6	0.45102%
8	0.19498%

Table 2.- Differents probability errors.

If continuous-uniform distribution probability is chosen instead beta distribution, the error probability goes up to 2.15392% (Note it is a low percentage too).

6. UTILITY AND REPERCUSSION OF MAKING ROUND THE TEMPLATES.

When you make round the template corners, slight repercussion appears over bounds. The roundness make a smaller contour. Then the relative controllability grows and this is translated into less strict bounds.

All QFT specifications can be quantified, then if the strictness of bounds is high, and it is impossible to get a design of the controller, we must relax specifications to achieve a design. As we had quantified, we know the new requirements for stability, disturbance rejection, etc. But there is a specification that is difficult to quantify: tracking specification. When you must relax tracking is very difficult to quantify this relaxation.

One way to quantify tracking specification is by means of template roundness. When we make round the template corners, the bounds go down in the Nichols chart; like when we relax specifications. Now we can quantify this relaxation with probability error.

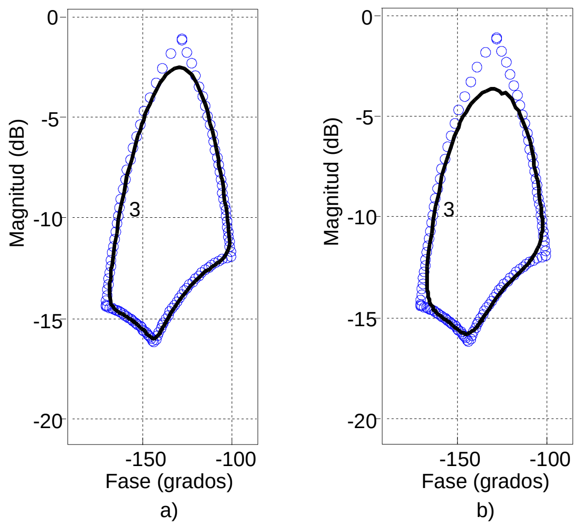

Figure 13.- Original and rounded templates for plant $P(s)=\frac{10}{s^3+as^2+bs+20}$ $a\in[3,8]; b\in[12,22]$ and $\omega=3$ rad/s . It was used (8) with $m_i=8$ in a) and $m_i=4$ in b).

Using the rounded templates shown in figure 13, the tracking bounds were calculated for everyone using the tracking boundaries: $ml=\frac{20}{0.1s^3+2.3s^2+15s+20}$ and $mu=\frac{100}{s^2+40s+100}$. The results are shown in figure 14.

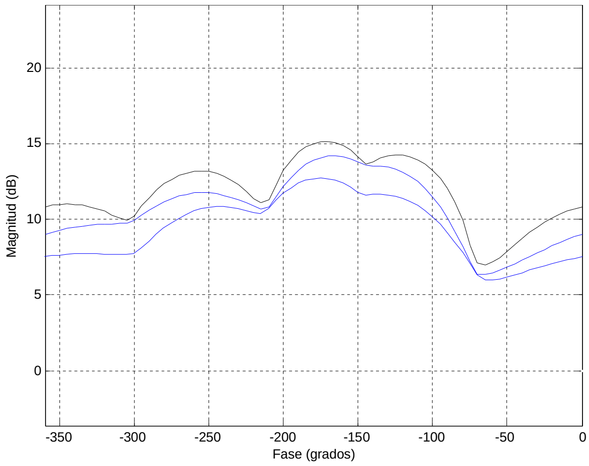

Figure 14.- Several bounds for the same QFT specification, but making round template corners.

In figure 14 are shown three bounds for the same specification and same plant. The difference between them it is the uncertainty. The most strict bound is achieved by means of template without roundness. The second most strict bound it is derived when we suppose a rounded template. In this case, it was chosen the template b) of figure 13. We can see that the difference is slight.

But, depending on probability distribution of parameters, the template area reduction can be done with other criteria. In the beta distribution of the last example we can observe that the *a* uncertain parameter can be reduced (assuming a small error) to $a\in[3, 7]$ (instead $a\in[3, 8]$) because the volume between $8<a<7$ is despised. It allows to get less strict bounds. If we take $a\in[3, 7]$ the probability errors are mentioned on table 2; on the other hand if we take $a\in[3, 7]$ the probability errors are:

n	p_{error}
4	3.3775%
6	2.4641%
8	2.2159%

Table 3.- Differents probability errors.

This was the option chosen to get the bound lesser strict than others in figure 14.

Then, we can relax the bounds without relax specifications. Thus, we now the plant percentage that keep specifications. Finally, we do not get a system that always keeps the original specifications (like when we relax ones), but we know the percentage that keeps them.

Other utility of making the templates round is as follows: When we make round a template, the corners are eliminated, and instead of them, curves arise. The obtained template is a soft Jordan curve (it has derivative in every point). This offers the possibility to describe template in an analytic way, as is exposed in Martín-Romero (2005).

In fact, this paper was motived by the necessity of obtaining a way to get analytic templates. If we do not make round off the edges, we can define an analytic template too, but the operations with them are more difficult, specially if you need to do the derivative, because it does not exist in some points or will be bad calculated. Work with rounded analytic templates allows to calculate templates in plants with high number of uncertainty parameters.

7. CONCLUSIONS.

A small study was presented about the use of probability in QFT. Some results about despise of template contour points were mentionated and they were shown by means of an example.

Two applications were also described from making round the template corners. First, the possibility to get more relaxed bounds without change specifications. Second, we can obtain analytic templates with better properties.

8. REFERENCES.

García-Sanz, M., (2004). .

Horowitz, I. (1991). Survey of Quantitative Feedback Theory (QFT). Int. J. Control, 53, Vol 2, pp. 255-291.

Martín-Romero, J. J. (2005). Plant templates with a high number of uncertainty parameters.